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A transcendental approach to Kollár’s injectivity theorem II

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Abstract. We treat a relative version of the main theorem in “A transcendental approach to Kollár’s injectivity theorem” (Osaka J. Math., to appear). More explicitly, we give a curvature condition that implies Kollár type cohomology injectivity theorems in the relative setting. To carry out this generalization, we use the Ohsawa–Takegoshi twisted version of Nakano’s identity.

1. Introduction

The following theorem is the main theorem of this paper, which is a relative version of the main theorem in [13]. It is a generalization of Kollár’s injectivity theorem (cf. [20, Theorem 2.2]). More precisely, it is a generalization of Enoki’s injectivity theorem, which is an analytic version of Kollár’s theorem (see [6, Theorem 0.2] and [13, Corollary 1.4]). We note that Kollár’s proofs and the approach of Esnault–Viehweg are geometric (cf. [20], [21, Chapters 9 and 10], and [7]) and are not related to curvature conditions. Therefore, we do not know the true relationship between Kollár’s injectivity theorem and Enoki’s one. All the terms in the statements will be defined in Section 2.

Theorem 1.1 (Main Theorem). *Let $f : X \rightarrow Y$ be a proper surjective morphism from a Kähler manifold X to a complex variety Y . Let (E, h_E) (resp. (L, h_L)) be a holomorphic vector (resp. line) bundle on X with a smooth hermitian metric h_E (resp. h_L). Let F be a holomorphic line bundle on X with a singular hermitian metric h_F . Assume the following conditions.*

- (i) *There exists a subvariety Z of X such that h_F is smooth on $X \setminus Z$.*
- (ii) *$\sqrt{-1}\Theta(F) \geq -\tilde{\gamma}$ in the sense of currents, where $\tilde{\gamma}$ is a smooth $(1, 1)$ -form on X .*
- (iii) *$\sqrt{-1}(\Theta(E) + \text{Id}_E \otimes \Theta(F)) \geq_{\text{Nak}} 0$ on $X \setminus Z$.*
- (iv) *$\sqrt{-1}(\Theta(E) + \text{Id}_E \otimes \Theta(F) - \varepsilon_0 \text{Id}_E \otimes \Theta(L)) \geq_{\text{Nak}} 0$ on $X \setminus Z$ for some positive constant ε_0 .*

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Here, $\geq_{\text{Nak}} 0$ means the Nakano semi-positivity. Let s be a nonzero holomorphic section of L . Then the multiplication homomorphism

$$\times_s : R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \rightarrow R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F) \otimes L)$$

is injective for every $q \geq 0$, where K_X is the canonical line bundle of X and $\mathcal{J}(h_F)$ is the multiplier ideal sheaf associated to the singular hermitian metric h_F of F . Note that \times_s is the sheaf homomorphism induced by the tensor product with s .

We note that Theorem 1.1 will be generalized slightly in Proposition 4.1 below. For the absolute case and the background of Kollár type cohomology injectivity theorems, see the introduction of [13]. The reader who reads Japanese may find [9] also useful. The essential part of Theorem 1.1 is contained in Ohsawa's injectivity theorem (see [27]). Our formulation is much more suitable for geometric applications than Ohsawa's (cf. [13, 4. Applications]). We note that the main ingredient of our proof is the Ohsawa–Takegoshi twisted version of Nakano's identity (cf. Proposition 2.20).

The next corollary directly follows from Theorem 1.1. It contains a generalization of the Grauert–Riemenschneider vanishing theorem.

Corollary 1.2 (Torsion-freeness). *Let $f : X \rightarrow Y$ be a proper surjective morphism from a Kähler manifold X to a complex variety Y . Let (E, h_E) (resp. (F, h_F)) be a holomorphic vector (resp. line) bundle on the manifold X with a smooth hermitian metric h_E (resp. a singular hermitian metric h_F). Assume the conditions (i), (ii) and (iii) in Theorem 1.1. Then, $R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F))$ is torsion-free for every $q \geq 0$. In particular,*

$$R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) = 0 \quad \text{for } q > \dim X - \dim Y.$$

We will describe the proof of Theorem 1.1 in Section 3, which may help the reader to understand [27].

From now on, we discuss various vanishing theorems as applications of Theorem 1.1.

Corollary 1.3 (Kawamata–Viehweg–Nadel type vanishing theorem). *Let $f : X \rightarrow Y$ be a proper surjective morphism from a complex manifold X to a complex variety Y . Let E be a Nakano semi-positive vector bundle on X and let \mathcal{L} be a holomorphic line bundle on X such that $\mathcal{L}^{\otimes m} \simeq \mathcal{M} \otimes \mathcal{O}_X(D)$ where m is a positive integer, \mathcal{M} is an f -nef-big line bundle, and D is an effective Cartier divisor on X . Then*

$$R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0 \quad \text{for every } q > 0$$

where $\mathcal{J} = \mathcal{J}(\frac{1}{m}D)$ is the multiplier ideal sheaf associated to $\frac{1}{m}D$.

In the minimal model program for projective morphisms between complex varieties, the Kawamata–Viehweg vanishing theorem plays crucial roles. It was first obtained by Nakayama (cf. [25, Theorem 3.7]).

Corollary 1.4 (Kawamata–Viehweg vanishing theorem for proper morphisms). *Let*

$$f : X \rightarrow Y$$

be a proper surjective morphism from a complex manifold X to a complex variety Y . Let H be a \mathbb{Q} -Cartier \mathbb{Q} -divisor on X such that $\text{Supp}\{H\}$ is a normal crossing divisor on X and that

$\mathcal{O}_X(mH)$ is an f -nef-big line bundle for some positive integer m . Then

$$R^q f_*(K_X \otimes \mathcal{O}_X(\lceil H \rceil)) = 0 \quad \text{for every } q > 0.$$

We can also prove a Kollár type vanishing theorem from Theorem 1.1. The proof of Corollary 1.5 is a routine work for experts.

Corollary 1.5 (Kollár type vanishing theorem). *Let $f : X \rightarrow Y$ be a proper surjective morphism from a Kähler manifold X to a complex variety Y . Let $g : Y \rightarrow Z$ be a projective morphism between complex varieties. Let E be a Nakano semi-positive vector bundle on X and let \mathcal{L} be a holomorphic line bundle on X such that $\mathcal{L}^{\otimes m} \simeq f^* \mathcal{N} \otimes \mathcal{O}_X(D)$ where m is a positive integer, \mathcal{N} is a g -nef-big line bundle, and D is an effective Cartier divisor on X . Then*

$$R^p g_* R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0$$

for every $p > 0$ and every $q \geq 0$ where $\mathcal{J} = \mathcal{J}(\frac{1}{m}D)$ is the multiplier ideal sheaf associated to $\frac{1}{m}D$.

All the statements in the introduction may look complicated. So it seems to be worth mentioning that the following well-known vanishing theorems easily follow from the main theorem, Theorem 1.1. The proofs of Corollary 1.6 and Corollary 1.7 explain one of the reasons why we think that injectivity theorems are generalizations of vanishing theorems (cf. [8], [10]) and that the formulation of Theorem 1.1 is useful for various applications.

Corollary 1.6 (Nakano vanishing theorem (cf. [4, (4.9)])). *Let X be a compact complex manifold and let E be a Nakano positive vector bundle on X . Then $H^q(X, K_X \otimes E) = 0$ for every $q > 0$.*

Proof. Since E is Nakano positive, $L = \det E$ is a positive line bundle. Therefore, X is projective and L is ample by Kodaira's embedding theorem. If ε is a small positive number, then

$$\sqrt{-1}(\Theta(E) - \varepsilon \operatorname{Id}_E \otimes \Theta(L)) \geq_{\text{Nak}} 0.$$

Thus, by using Theorem 1.1, $H^q(X, K_X \otimes E)$ can be embedded into $H^q(X, K_X \otimes E \otimes L^{\otimes m})$ for every sufficiently large positive integer m . By Serre's vanishing theorem,

$$H^q(X, K_X \otimes E) = 0 \quad \text{for every } q > 0. \quad \square$$

Corollary 1.7 (Kawamata–Viehweg vanishing theorem). *Let X be a smooth projective variety and let L be a nef and big line bundle on X . Then*

$$H^q(X, K_X \otimes L) = 0 \quad \text{for every } q > 0.$$

Proof. By Kodaira's lemma, we can write $L^{\otimes m} \simeq A \otimes \mathcal{O}_X(D)$ such that m is a positive integer, A is an ample line bundle, and D is an effective Cartier divisor on X satisfying $\mathcal{J}(\frac{1}{m}D) = \mathcal{O}_X$. Let h_D be the singular hermitian metric of $\mathcal{O}_X(D)$ naturally associated to D (cf. [13, Example 2.3]) and let h_A be a smooth hermitian metric of A whose curvature is positive. We put

$$h_L = h_A^{\frac{1}{m}} h_D^{\frac{1}{m}}.$$

Then $\sqrt{-1}\Theta(L) \geq 0$ in the sense of currents, h_L is smooth on $X \setminus D$, and

$$\sqrt{-1}(\Theta(L) - \varepsilon\Theta(A)) \geq 0$$

on $X \setminus D$ for $0 < \varepsilon \ll 1$. Therefore, we have inclusions

$$H^q(X, K_X \otimes L) \subset H^q(X, K_X \otimes L \otimes A^{\otimes l})$$

for every q and every sufficiently large positive integer l by Theorem 1.1. Thus

$$H^q(X, K_X \otimes L) = 0 \quad \text{for every } q > 0$$

by Serre's vanishing theorem. □

For related topics, see [25, Section 3], [26, II, Section 5.c, V, Section 3], and [30]. We think that one of the most important open problems on vanishing theorems for the minimal model program is to prove the results in [12, Sections 6 and 8] and [14, Chapter 2] for projective morphisms between complex varieties. The following problem seems to be a first step.

Problem 1.8. Let $f : X \rightarrow Y$ be a projective morphism from a complex manifold X onto a complex variety Y . Let D be a simple normal crossing divisor on X . Let L be a π -semi-ample Cartier divisor on X . Let s be a nonzero holomorphic section of kL for some positive integer k . Assume that $(s = 0)$ contains no lc centers of (X, D) . Then

$$\times s : R^q \pi_*(K_X \otimes \mathcal{O}_X(D + mL)) \rightarrow R^q \pi_*(K_X \otimes \mathcal{O}_X(D + (m + k)L))$$

is injective for every q and every positive integer m . Note that $W \subset X$ is an lc center of (X, D) if and only if W is an irreducible component of $D_{i_1} \cap \cdots \cap D_{i_l}$ for some $\{i_1, \dots, i_l\} \subset I$, where $D = \sum_{i \in I} D_i$ is the irreducible decomposition.

The statement of Problem 1.8 holds true when Y is a point (see, for example, [11], [12], [14], and [16]). We do not know if the statement of Problem 1.8 holds true or not under the weaker assumption that $\mathcal{O}_X(L)$ is semi-positive even when Y is a point. When $D = 0$, the statement of Problem 1.8 is a special case of Theorem 1.1. In Problem 1.8, it may be sufficient to assume that f is a Kähler morphism. For the solution of Problem 1.8, we cannot directly use the arguments in this paper because the L^2 -method does not work for log canonical pairs. We note that the arguments in [11], [12], [14], [15], and [16] are geometric.

We summarize the contents of this paper. In Section 2, we collect basic definitions and results in the algebraic and analytic geometries. In this section, we discuss the Ohsawa–Takegoshi twisted version of Nakano's identity. It is a key ingredient of the proof of the main theorem, Theorem 1.1. Section 3 is devoted to the proof of the main theorem, Theorem 1.1. In Section 4, we discuss the proofs of the corollaries in Section 1 and some applications. In the final section, Section 5, we discuss various examples of nef, semi-positive, and semi-ample line bundles. It is very important to understand the differences in the notion of semi-ample, semi-positive, and nef line bundles.

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2. Preliminaries

In this section, we collect basic definitions and results in the algebraic and analytic geometries.

2.1 (Projective morphisms). For details of *projective morphisms* of complex varieties and ample line bundles, see, for example, [25, Section 1] and [26, II. 1.10. Definition, Remark].

2.2 (Big line bundles). In this paper, we will freely use *Iitaka's D-dimension* κ , the *numerical D-dimension* ν , and so on, for algebraic varieties.

Let us recall the definition of *f-nef-big line bundles* for proper morphisms between complex varieties, which we need in corollaries in Section 1.

Definition 2.3 (cf. [25, Definition]). Let $f : X \rightarrow Y$ be a proper surjective morphism from a complex variety X onto a complex variety Y . Let L be a line bundle on X . Then L is called *f-big* if the relative Iitaka D -dimension $\kappa(X/Y, L) = \dim X - \dim Y$. Furthermore, if $L \cdot C \geq 0$ for every irreducible curve C such that $f(C)$ is a point, then L is called *f-nef-big*.

2.4 (\mathbb{Q} -divisors). Let $D = \sum_i d_i D_i$ be a \mathbb{Q} -divisor on a normal complex variety X where D_i is a prime divisor for every i and $D_i \neq D_j$ for $i \neq j$. Then we define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. the *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where for every real number x , $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$.

2.5 (Singular hermitian metric). Let L be a holomorphic line bundle on a complex manifold X .

Definition 2.6 (Singular hermitian metric). A *singular hermitian metric* on L is a metric which is given in any trivialization $\theta : L|_\Omega \simeq \Omega \times \mathbb{C}$ by

$$\|\xi\| = |\theta(\xi)|e^{-\varphi(x)}, \quad x \in \Omega, \quad \xi \in L_x,$$

where $\varphi \in L^1_{\text{loc}}(\Omega)$ is an arbitrary function, called the *weight* of the metric with respect to the trivialization θ . Here, $L^1_{\text{loc}}(\Omega)$ is the space of the locally integrable functions on Ω .

2.7 (Multiplier ideal sheaf). The notion of multiplier ideal sheaves introduced by Nadel is very important. First, we recall the notion of (quasi-)plurisubharmonic functions.

Definition 2.8 (Plurisubharmonic function). A function $u : \Omega \rightarrow [-\infty, \infty)$ defined on an open set $\Omega \subset \mathbb{C}^n$ is called *plurisubharmonic* (psh, for short) if

- (1) u is upper semi-continuous, and
- (2) for every complex line $L \subset \mathbb{C}^n$, $u|_{\Omega \cap L}$ is subharmonic on $\Omega \cap L$, that is, for every $a \in \Omega$ and $\xi \in \mathbb{C}^n$ satisfying $|\xi| < d(a, \Omega^c)$, the function u satisfies the mean inequality

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\xi) d\theta.$$

Let X be an n -dimensional complex manifold. A function $\varphi : X \rightarrow [-\infty, \infty)$ is said to be *plurisubharmonic* (psh, for short) if there exists an open cover $X = \bigcup_{i \in I} U_i$ such that $\varphi|_{U_i}$ is

plurisubharmonic on $U_i (\subset \mathbb{C}^n)$ for every i . A *smooth strictly plurisubharmonic function* ψ on X is a smooth function on X such that $\sqrt{-1}\partial\bar{\partial}\psi$ is a positive definite smooth $(1, 1)$ -form.

Definition 2.9. A *quasi-plurisubharmonic* (quasi-psh, for short) function is a function φ which is locally equal to the sum of a psh function and of a smooth function.

Next, we define multiplier ideal sheaves.

Definition 2.10 (Multiplier ideal sheaf). If φ is a quasi-psh function on a complex manifold X , the *multiplier ideal sheaf* $\mathcal{J}(\varphi) \subset \mathcal{O}_X$ is defined by

$$\Gamma(U, \mathcal{J}(\varphi)) = \{f \in \mathcal{O}_X(U) \mid |f|^2 e^{-2\varphi} \in L^1_{\text{loc}}(U)\}$$

for every open set $U \subset X$. Then it is known that $\mathcal{J}(\varphi)$ is a coherent ideal sheaf of \mathcal{O}_X . See, for example, [4, (5.7) Proposition].

Finally, we note the definition of $\mathcal{J}(h_F)$ in Theorem 1.1.

Remark 2.11. In Theorem 1.1, the curvature $\sqrt{-1}\Theta(F)$ of (F, h_F) is expressed by $\sqrt{-1}\partial\bar{\partial}\log h_F$. Therefore, if we write $h_F = e^{-2\varphi}$ locally, then

$$\sqrt{-1}\Theta(F) = \sqrt{-1}\partial\bar{\partial}\log h_F = 2\partial\bar{\partial}\varphi.$$

By the assumption (ii) in Theorem 1.1, we may assume that the weight φ of the singular hermitian metric h_F is a quasi-psh function on any trivialization. So, we can define multiplier ideal sheaves locally and check that they are independent of trivializations. Thus, we can define the multiplier ideal sheaf globally and denote it by $\mathcal{J}(h_F)$, which is an abuse of notation. It is a coherent ideal sheaf on X .

2.12 (Kähler geometry). We collect the basic notion and results of the hermitian and Kähler geometries (see also [4]).

Definition 2.13 (Chern connection and its curvature form). Let X be a complex hermitian manifold and let (E, h) be a holomorphic hermitian vector bundle on X . Then there exists the *Chern connection* $D = D_{(E, h)}$, which can be split in a unique way as a sum of a $(1, 0)$ - and of a $(0, 1)$ -connection,

$$D = D'_{(E, h)} + D''_{(E, h)}.$$

By the definition of the Chern connection, $D'' = D''_{(E, h)} = \bar{\partial}$. We obtain the *curvature form*

$$\Theta(E) = \Theta_{(E, h)} = \Theta_h := D_{(E, h)}^2.$$

The subscripts might be suppressed if there is no danger of confusion.

Definition 2.14 (Inner product). Let X be an n -dimensional complex manifold with the hermitian metric g . We denote by ω the *fundamental form* of g . Let (E, h) be a hermitian vector bundle on X and let u, v be E -valued (p, q) -forms with measurable coefficients. We set

$$\|u\|^2 = \int_X |u|^2 dV_\omega, \quad \langle\langle u, v \rangle\rangle = \int_X \langle u, v \rangle dV_\omega,$$

where $|u|$ is the pointwise norm induced by g and h on $\Lambda^{p,q} T_X^* \otimes E$, and $dV_\omega = \frac{1}{n!} \omega^n$. More explicitly, $\langle u, v \rangle dV_\omega = {}^t u \wedge H^* \bar{v}$, where ${}^t u$ is the transposed matrix of u , $*$ is the Hodge star operator relative to ω , and H is the (local) matrix representation of h . When we need to emphasize the metrics, we write $|u|_{g,h}$, and so on.

Lemma 2.15 (Adjoint). *Let $\theta \in C^{s,t}(X)$ where $C^{s,t}(X)$ denotes the space of smooth (s, t) -forms on X . Let θ^* be the adjoint operator of $\theta \wedge \cdot$ with respect to $\langle \cdot, \cdot \rangle$, that is,*

$$\langle \theta \wedge v, u \rangle = \langle v, \theta^* u \rangle$$

for every $u \in C^{p,q}(X, E)$ and every $v \in C^{p-s, q-t}(X, E)$. Note that we denote by $C^{p,q}(X, E)$ (resp. $C^{p-s, q-t}(X, E)$) the space of smooth E -valued (p, q) -forms (resp. $(p-s, q-t)$ -forms) on X . Then

$$\theta^* = (-1)^{(p+q)(s+t+1)} * \bar{\theta} *$$

on $C^{p,q}(X, E)$. In particular, if θ is a 1-form, then $\theta^* = * \bar{\theta} *$.

Proof. We take $u \in C^{p,q}(X, E)$ and $v \in C^{p-s, q-t}(X, E)$. Then

$$\langle v, \theta^* u \rangle dV_\omega = \langle \theta \wedge v, u \rangle dV_\omega$$

by the definition of θ^* . Let H be the local matrix representation of h . Then we have

$$\begin{aligned} \langle \theta \wedge v, u \rangle dV_\omega &= {}^t(\theta \wedge v) H^* \bar{u} \\ &= (-1)^{(p+q-s-t)(s+t)} ({}^t v) H \theta^* \bar{u} \\ &= (-1)^{(p+q-s-t)(s+t) + (2n-p-q+s+t)} ({}^t v) H * \bar{\theta} * u \\ &= (-1)^{(p+q)(s+t+1)} ({}^t v) H * \bar{\theta} * u. \end{aligned}$$

Therefore, $\theta^* u = (-1)^{(p+q)(s+t+1)} * \bar{\theta} * u$. □

Let $L_{(2)}^{p,q}(X, E) (= L_{(2)}^{p,q}(X, (E, h)))$ denote the space of square integrable E -valued (p, q) -forms on X . The inner product was defined in Definition 2.14. When we emphasize the metrics, we write $L_{(2)}^{p,q}(X, E)_{g,h}$, where g (resp. h) is the hermitian metric of X (resp. E). As usual one can view D' and D'' as closed and densely defined operators on the Hilbert space $L_{(2)}^{p,q}(X, E)$. The formal adjoints D'^* , D''^* also have closed extensions in the sense of distributions, which do not necessarily coincide with the Hilbert space adjoints in the sense of von Neumann, since the latter ones may have strictly smaller domains. It is well known, however, that the domains coincide if the hermitian metric of X is complete. See Lemma 2.16 below.

Lemma 2.16 (Density Lemma). *Let X be a complex manifold with the complete hermitian metric g and let (E, h) be a holomorphic hermitian vector bundle on X . Then $C_0^{p,q}(X, E)$ is dense in $\text{Dom}(\bar{\partial}) \cap \text{Dom}(D''^*_{(E,h)})$ with respect to the graph norm*

$$\|v\| + \|\bar{\partial} v\| + \|D''^*_{(E,h)} v\|,$$

where $\text{Dom}(\bar{\partial})$ (resp. $\text{Dom}(D''^*_{(E,h)})$) is the domain of $\bar{\partial}$ (resp. $D''^*_{(E,h)}$). Here $C_0^{p,q}(X, E)$ is the space of smooth E -valued (p, q) -forms with compact supports on X .

Suppose that (E, h) is a holomorphic hermitian vector bundle and that (e_λ) is a holomorphic frame for E over some open set U . Then the metric h is given by the $r \times r$ hermitian

matrix $H = (h_{\lambda\mu})$, where $h_{\lambda\mu} = h(e_\lambda, e_\mu)$ and $r = \text{rank } E$. Then we have $h(u, v) = {}^t u H \bar{v}$ on U for smooth sections u, v of $E|_U$. This implies that

$$h(u, v) = \sum_{\lambda, \mu} u_\lambda h_{\lambda\mu} \bar{v}_\mu$$

for $u = \sum e_i u_i$ and $v = \sum e_j v_j$. Then we obtain that

$$\sqrt{-1}\Theta_h(E) = \sqrt{-1}\bar{\partial}(\bar{H}^{-1}\partial\bar{H}) \quad \text{and} \quad {}^t(\sqrt{-1}{}^t\Theta_h(E)H) = \sqrt{-1}{}^t\Theta_h(E)H$$

on U . Let $C^{p,q}(X, E)$ (resp. $C_0^{p,q}(X, E)$) denote the space of smooth E -valued (p, q) -forms (resp. smooth E -valued (p, q) -forms with compact supports) on X . Define $\{u, v\} = {}^t u \wedge H \bar{v}$ for $u \in C^{p,q}(X, E)$ and $v \in C^{r,s}(X, E)$, where ${}^t u$ is the transposed matrix of u . We will use $\{\cdot, \cdot\}$ in Section 3.

Definition 2.17 (Nakano positivity and semi-positivity). Let (E, h) be a holomorphic vector bundle on a complex manifold X with a smooth hermitian metric h . Let Ξ denote a $\text{Hom}(E, E)$ -valued $(1, 1)$ -form such that ${}^t(\bar{\partial}\Xi) = {}^t\Xi h$. Then Ξ is said to be *Nakano positive* (resp. *Nakano semi-positive*) if the hermitian form on $T_X \otimes E$ associated to ${}^t\Xi h$ is positive definite (resp. semi-definite). We write $\Xi >_{\text{Nak}} 0$ (resp. $\Xi \geq_{\text{Nak}} 0$). We note that $\Xi_1 >_{\text{Nak}} \Xi_2$ (resp. $\Xi_1 \geq_{\text{Nak}} \Xi_2$) means that $\Xi_1 - \Xi_2 >_{\text{Nak}} 0$ (resp. $\geq_{\text{Nak}} 0$). A holomorphic vector bundle (E, h) is said to be *Nakano positive* (resp. *semi-positive*) if $\sqrt{-1}\Theta(E) >_{\text{Nak}} 0$ (resp. $\geq_{\text{Nak}} 0$). We usually omit “Nakano” when E is a line bundle. We often simply say that a holomorphic line bundle L is *semi-positive* if there exists a smooth hermitian metric h_L on L such that $\sqrt{-1}\Theta(L) \geq 0$.

The space of harmonic forms will play important roles in the proof of Theorem 1.1. See also the introduction of [13].

Definition 2.18 (Harmonic forms). Let X be an n -dimensional complete Kähler manifold with a complete Kähler metric g . Let (E, h_E) be a holomorphic hermitian vector bundle on X . We put

$$\mathcal{H}^{p,q}(X, (E, h_E))_g = \{u \in L_{(2)}^{p,q}(X, E) \mid \bar{\partial}u = 0 \text{ and } D''_{(E, h_E)}^* u = 0\}.$$

Note that $\mathcal{H}^{p,q}(X, (E, h_E))_g \subset C^{p,q}(X, E)$ by the regularization theorem for elliptic partial differential equations of second order.

2.19 (Ohsawa–Takegoshi twist). The following formula is a *twisted* version of Nakano’s identity, which is now well known to the experts.

Proposition 2.20 (Ohsawa–Takegoshi twist). Let (E, h) be a holomorphic hermitian vector bundle on an n -dimensional Kähler manifold X . Let η be any smooth positive function on X . Then, for every $u \in C_0^{n,q}(X, E)$, the equality

$$\begin{aligned} (\spadesuit) \quad & \|\sqrt{\eta}D''_{(E, h)}^* u\|^2 + \|\sqrt{\eta}\bar{\partial}u\|^2 - \|\sqrt{\eta}D'^* u\|^2 \\ & = \langle \sqrt{-1}(\eta\Theta_h - \text{Id}_E \otimes \bar{\partial}\bar{\partial}\eta)\Lambda u, u \rangle + 2\text{Re} \langle \bar{\partial}\eta \wedge D''_{(E, h)}^* u, u \rangle \end{aligned}$$

holds true. Here, we denote by Λ the adjoint operator of $\omega \wedge \cdot$. Note that $D'' = D''_{(E, h)} = \bar{\partial}$ and D'^* are independent of the hermitian metric h .

Sketch of the proof. We quickly review the proof of this proposition for the reader's convenience. If A, B are the endomorphisms of pure degree of the graded module $C^{\bullet, \bullet}(X, E)$, their *graded Lie bracket* is defined by

$$[A, B] = AB - (-1)^{\deg A \deg B} BA.$$

Let

$$\Delta' = D'D'^* + D'^*D' \quad \text{and} \quad \Delta'' = D''D''^* + D''^*D''$$

be the complex Laplace operators. Then it is well known that

$$\Delta'' = \Delta' + [\sqrt{-1}\Theta(E), \Lambda],$$

which is sometimes called Nakano's identity. Let us consider the *twisted* Laplace operators

$$D'\eta D'^* + D'^*\eta D' = \eta\Delta' + (\partial\eta)D'^* - (\partial\eta)^*D',$$

and

$$D''\eta D''^* + D''^*\eta D'' = \eta\Delta'' + (\bar{\partial}\eta)D''^* - (\bar{\partial}\eta)^*D''.$$

On the other hand, we can easily check that

$$[\sqrt{-1}\partial\bar{\partial}\eta, \Lambda] = [D'', (\bar{\partial}\eta)^*] + [D'^*, \partial\eta]$$

by $(\bar{\partial}\eta)^* = -\sqrt{-1}[\partial\eta, \Lambda]$ and $D'^* = -\sqrt{-1}[\bar{\partial}, \Lambda]$. Combining these equalities, we find

$$\begin{aligned} D''\eta D''^* + D''^*\eta D'' - D'\eta D'^* - D'^*\eta D' + [\sqrt{-1}\partial\bar{\partial}\eta, \Lambda] \\ = \eta[\sqrt{-1}\Theta(E), \Lambda] + (\bar{\partial}\eta)D''^* + D''(\bar{\partial}\eta)^* + (\partial\eta)^*D' + D'^*(\partial\eta). \end{aligned}$$

Apply this identity to a form $u \in C_0^{n,q}(X, E)$ and take the inner product with u . Then we obtain the desired formula. \square

The next proposition is [27, Lemma 2.1]. The proof is a routine work. It easily follows from Lemmas 2.16, 2.22, and 2.23.

Proposition 2.21. *Fix a complete Kähler metric g on X . We put*

$$D^{n,q} = \{u \in L_{(2)}^{n,q}(X, E) \mid \bar{\partial}u \in L_{(2)}^{n,q+1}(X, E) \text{ and } D''^*u \in L_{(2)}^{n,q-1}(X, E)\},$$

that is, $D^{n,q} = \text{Dom}(\bar{\partial}) \cap \text{Dom}(D''^) \subset L_{(2)}^{n,q}(X, E)$. Suppose that η is bounded and that there exists a constant $\varepsilon > 0$ such that*

$$\sqrt{-1}(\eta\Theta_h - \text{Id}_E \otimes \partial\bar{\partial}\eta - \varepsilon \text{Id}_E \otimes \partial\eta \wedge \bar{\partial}\eta) \geq_{\text{Nak}} 0$$

holds everywhere. Then equality (\spadesuit) in Proposition 2.20 holds for all $u \in D^{n,q}$.

Lemma 2.22. *For every $u \in C^{n,q}(X, E)$ and any positive real number δ , we have*

$$\begin{aligned} |2 \operatorname{Re} \langle \bar{\partial}\eta \wedge D''^*_{(E,h)} u, u \rangle| &= |2 \operatorname{Re} \langle D''^*_{(E,h)} u, (\bar{\partial}\eta)^* u \rangle| \\ &\leq \frac{1}{\delta} \|D''^*_{(E,h)} u\|^2 + \delta \|(\bar{\partial}\eta)^* u\|^2 \end{aligned}$$

and

$$\begin{aligned}\|(\bar{\partial}\eta)^*u\|^2 &= \langle (\bar{\partial}\eta)^*u, (\bar{\partial}\eta)^*u \rangle \\ &= \langle \sqrt{-1}\bar{\partial}\eta \wedge \bar{\partial}\eta \wedge u, u \rangle\end{aligned}$$

since $(\bar{\partial}\eta)^*u = -\sqrt{-1}\bar{\partial}\eta \wedge u$ for $u \in C^{n,q}(X, E)$. Note that $(\bar{\partial}\eta)^*$ is the adjoint operator of $\bar{\partial}\eta \wedge \cdot$ relative to the inner product $\langle \cdot, \cdot \rangle$.

By combining Proposition 2.20 with Lemma 2.22, we obtain the next lemma.

Lemma 2.23. *We use the same notation as in Proposition 2.20. Assume that*

$$\sqrt{-1}(\eta\Theta_h - \text{Id}_E \otimes \bar{\partial}\bar{\partial}\eta - \varepsilon \text{Id}_E \otimes \bar{\partial}\eta \wedge \bar{\partial}\eta) \geq_{\text{Nak}} 0$$

holds everywhere for some positive constant ε . Then, for every $u \in C_0^{n,q}(X, E)$, we have

$$\|\sqrt{\eta}D'^*u\|^2 \leq \|\sqrt{\eta}D''^*_{(E,h)}u\|^2 + \|\sqrt{\eta}\bar{\partial}u\|^2 + \frac{1}{\varepsilon}\|\sqrt{\eta}D''^*_{(E,h)}u\|^2,$$

and

$$\|\sqrt{\eta}D''^*_{(E,h)}u\|^2 + \|\sqrt{\eta}\bar{\partial}u\|^2 - \|\sqrt{\eta}D'^*u\|^2 + \frac{1}{\delta}\|\sqrt{\eta}D''^*_{(E,h)}u\|^2 \geq (\varepsilon - \delta)\|(\bar{\partial}\eta)^*u\|^2$$

for any positive real number δ .

We close this section by the following remark on [30].

Remark 2.24. By Proposition 2.21, we can prove [30, Theorem 3.4(ii)] under the slightly weaker assumption that φ is a *bounded* smooth psh function on M . We do not have to assume that $|d\varphi|$ is bounded on M . For the notations, see [30]. In this case, there are positive constants C_1 and C_2 such that $\varphi + C_1 > 0$ on M and $C_2 - (\varphi + C_1)^2 > 0$ on M . We can use Proposition 2.21 (and Lemmas 2.22, 2.23) for $\eta := C_2 - (\varphi + C_1)^2$ and $\varepsilon := \frac{1}{2C_2} > 0$. Then we obtain

$$(\bar{\partial}\varphi)^*u = 0 \quad \text{and} \quad \langle \sqrt{-1}\bar{\partial}\bar{\partial}\varphi \wedge u, u \rangle_h = 0.$$

3. Proof of the main theorem

In this section, we prove Theorem 1.1. So, we freely use the notation in Theorem 1.1. Let $W \Subset Y$ be any Stein open subset. We put $V = f^{-1}(W)$. Then V is a holomorphically convex weakly 1-complete Kähler manifold (see Remark 3.1 below). Let \mathcal{F} be a coherent sheaf on V . Then

$$f^* : H^q(V, \mathcal{F}) \rightarrow \Gamma(W, R^q f_* \mathcal{F})$$

is an isomorphism of topological vector spaces for every q (cf. [28, Lemma II.1.]). In particular, $H^q(V, \mathcal{F})$ is a separated topological space. Therefore, to prove Theorem 1.1, it is sufficient to show that

$$\times s : H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F)) \rightarrow H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F) \otimes L)$$

is injective for every $q \geq 0$.

Remark 3.1. A weakly 1-complete manifold X is called a *weakly pseudoconvex manifold* in [4]. A weakly 1-complete manifold is a complex manifold equipped with a *smooth* plurisubharmonic exhaustion function. More explicitly, there exists a smooth plurisubharmonic function φ on X such that $X_c = \{x \in X \mid \varphi(x) < c\}$ is relatively compact in X for every c .

We define bounded smooth functions from the given nonzero holomorphic section s of L .

Definition 3.2. Take a smooth plurisubharmonic exhaustion function φ on V . Without loss of generality, we can assume that $\min_{x \in V} \varphi(x) = 0$. Let s be a holomorphic section of L . Let $|s|$ be the pointwise norm of s with respect to the fiber metric h_L . Let $\lambda : [0, \infty) \rightarrow [0, \infty)$ be a smooth convex increasing function such that $5|s|^2 < e^{\lambda(\varphi)}$. Thus,

$$|s|_{\lambda(\varphi)}^2 < \frac{1}{5} < \frac{1}{4},$$

where $|s|_{\lambda(\varphi)}$ is the pointwise norm of s with respect to the fiber metric $h_L e^{-\lambda(\varphi)}$. We put $\mu(x) := \lambda(x) + x$. Obviously, μ is also a smooth convex increasing function.

Definition 3.3. We put $\chi(t) = t - \log(-t)$ for $t < 0$. We define

$$\begin{aligned} \sigma_{\varepsilon, \lambda} &= \log(|s|_{\lambda(\varphi)}^2 + \varepsilon), \\ \eta_{\varepsilon, \lambda} &= \frac{1}{\varepsilon} - \chi(\sigma_{\varepsilon, \lambda}) = -\log(|s|_{\lambda(\varphi)}^2 + \varepsilon) + \log(-\log(|s|_{\lambda(\varphi)}^2 + \varepsilon)) + \frac{1}{\varepsilon} \end{aligned}$$

for a small positive real number ε . We can also define $\sigma_{\varepsilon, \mu}$ and $\eta_{\varepsilon, \mu}$ similarly. Note that $\eta_{\varepsilon, \lambda}$ and $\eta_{\varepsilon, \mu}$ are smooth bounded functions on V with $\eta_{\varepsilon, \mu} \geq \eta_{\varepsilon, \lambda} > \frac{1}{\varepsilon}$. The subscripts λ , μ , and ε might be suppressed if there is no danger of confusion.

We note the following obvious remark before we start various calculations.

Remark 3.4. We note that $e < 2\sqrt{2}$. Thus, $\frac{3}{2} \log 2 > 1$. Therefore,

$$\sigma_{\varepsilon, \mu} \leq \sigma_{\varepsilon, \lambda} < 2 \log \frac{1}{2} < -\frac{4}{3}$$

if ε is small since $|s|_{\mu(\varphi)}^2 + \varepsilon \leq |s|_{\lambda(\varphi)}^2 + \varepsilon < \frac{1}{4}$ by $|s|_{\lambda(\varphi)}^2 < \frac{1}{5}$. Of course,

$$\log(-\sigma_{\varepsilon, \mu}) \geq \log(-\sigma_{\varepsilon, \lambda}) > \log \frac{4}{3} > 0.$$

We have $\chi'(t) = 1 - \frac{1}{t}$ and $\chi''(t) = \frac{1}{t^2}$. Thus,

$$1 < \chi'(\sigma_{\varepsilon, \mu}) \leq \chi'(\sigma_{\varepsilon, \lambda}) = 1 + \frac{1}{(-\sigma_{\varepsilon, \lambda})} < \frac{7}{4}.$$

We also note that $-\chi(\sigma_{\varepsilon, \mu}) \geq -\chi(\sigma_{\varepsilon, \lambda}) \geq \frac{4}{3} + \log \frac{4}{3} > 0$.

3.5 (Basic calculations). We calculate various differentials of $\eta_{\varepsilon, \lambda}$. The same arguments work for $\eta_{\varepsilon, \mu}$.

Definition 3.6. Let $u \in C^{p, q}(V, L)$ and $v \in C^{r, s}(V, L)$. We define

$$\{u, v\}_{\lambda(\varphi)} = u \wedge H_L e^{-\lambda(\varphi)} \bar{v},$$

where H_L is the local matrix representation of h_L .

We have

$$\partial\sigma_{\varepsilon,\lambda} = \frac{\{D's, s\}_{\lambda(\varphi)}}{|s|_{\lambda(\varphi)}^2 + \varepsilon}.$$

In the above equation, D' is the $(1, 0)$ part of the Chern connection of $L' = (L, h_L e^{-\lambda(\varphi)})$, that is, $D' = D'_{(L, h_L e^{-\lambda(\varphi)})}$. Thus, $\Theta(L') = \Theta(L) + \partial\bar{\partial}\lambda(\varphi)$. We obtain the following equation by the direct computation:

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\sigma_{\varepsilon,\lambda} = & -\frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|_{\lambda(\varphi)}^2 + \varepsilon} + \frac{\sqrt{-1}\{D's, D's\}_{\lambda(\varphi)}}{|s|_{\lambda(\varphi)}^2 + \varepsilon} \\ & - \frac{\sqrt{-1}\{D's, s\}_{\lambda(\varphi)} \wedge \{s, D's\}_{\lambda(\varphi)}}{(|s|_{\lambda(\varphi)}^2 + \varepsilon)^2}, \end{aligned}$$

where $L' = (L, h_L e^{-\lambda(\varphi)})$. Since L is a line bundle, we have

$$\sqrt{-1}\{D's, D's\}_{\lambda(\varphi)}|s|_{\lambda(\varphi)}^2 = \sqrt{-1}\{D's, s\}_{\lambda(\varphi)} \wedge \{s, D's\}_{\lambda(\varphi)}.$$

Substituting it into the above equation, we obtain

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\sigma_{\varepsilon,\lambda} = & \frac{\varepsilon}{|s|_{\lambda(\varphi)}^2(|s|_{\lambda(\varphi)}^2 + \varepsilon)^2} \sqrt{-1}\{D's, s\}_{\lambda(\varphi)} \wedge \{s, D's\}_{\lambda(\varphi)} \\ & - \frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|_{\lambda(\varphi)}^2 + \varepsilon} \\ = & \frac{\varepsilon}{|s|_{\lambda(\varphi)}^2} \sqrt{-1}\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} - \frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|_{\lambda(\varphi)}^2 + \varepsilon}. \end{aligned}$$

Lemma 3.7. *The next equations easily follow from the definition:*

$$\begin{aligned} \partial\eta_{\varepsilon,\lambda} &= -\chi'(\sigma_{\varepsilon,\lambda})\partial\sigma_{\varepsilon,\lambda}, \\ \bar{\partial}\eta_{\varepsilon,\lambda} &= -\chi'(\sigma_{\varepsilon,\lambda})\bar{\partial}\sigma_{\varepsilon,\lambda}, \\ \partial\bar{\partial}\eta_{\varepsilon,\lambda} &= -\chi''(\sigma_{\varepsilon,\lambda})\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} - \chi'(\sigma_{\varepsilon,\lambda})\partial\bar{\partial}\sigma_{\varepsilon,\lambda}. \end{aligned}$$

Combining the above equalities, we have

$$\begin{aligned} -\sqrt{-1}\partial\bar{\partial}\eta_{\varepsilon,\lambda} &= \chi'(\sigma_{\varepsilon,\lambda})\sqrt{-1}\partial\bar{\partial}\sigma_{\varepsilon,\lambda} + \chi''(\sigma_{\varepsilon,\lambda})\sqrt{-1}\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} \\ &= \frac{\varepsilon\chi'(\sigma_{\varepsilon,\lambda})}{|s|_{\lambda(\varphi)}^2} \sqrt{-1}\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} + \chi''(\sigma_{\varepsilon,\lambda})\sqrt{-1}\partial\sigma_{\varepsilon,\lambda} \wedge \bar{\partial}\sigma_{\varepsilon,\lambda} \\ &\quad - \chi'(\sigma_{\varepsilon,\lambda}) \frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|_{\lambda(\varphi)}^2 + \varepsilon} \\ &= \left(\frac{\varepsilon}{\chi'(\sigma_{\varepsilon,\lambda})|s|_{\lambda(\varphi)}^2} + \frac{\chi''(\sigma_{\varepsilon,\lambda})}{\chi'(\sigma_{\varepsilon,\lambda})^2} \right) \sqrt{-1}\partial\eta_{\varepsilon,\lambda} \wedge \bar{\partial}\eta_{\varepsilon,\lambda} \\ &\quad - \chi'(\sigma_{\varepsilon,\lambda}) \frac{\{\sqrt{-1}\Theta(L')s, s\}_{\lambda(\varphi)}}{|s|_{\lambda(\varphi)}^2 + \varepsilon}. \end{aligned}$$

Lemma 3.8. *We have the following inequality:*

$$\frac{\chi''(\sigma_{\varepsilon,\lambda})}{\chi'(\sigma_{\varepsilon,\lambda})^2} \geq \eta_{\varepsilon,\lambda}^{-2}.$$

Proof. By definition, it is easy to see that

$$\frac{\chi'(\sigma_{\varepsilon,\lambda})}{\chi''(\sigma_{\varepsilon,\lambda})} = \frac{(1 - \frac{1}{\sigma_{\varepsilon,\lambda}})^2}{\frac{1}{\sigma_{\varepsilon,\lambda}^2}} = (\sigma_{\varepsilon,\lambda} - 1)^2.$$

On the other hand, $\eta_{\varepsilon,\lambda} = \frac{1}{\varepsilon} - \sigma_{\varepsilon,\lambda} + \log(-\sigma_{\varepsilon,\lambda}) > 1 - \sigma_{\varepsilon,\lambda}$. Thus, we obtain the desired inequality. \square

By the above calculations, we obtain a very important inequality.

Proposition 3.9. *Under the curvature conditions*

$$\sqrt{-1}(\Theta(E) + \text{Id}_E \otimes \Theta(F)) \geq_{\text{Nak}} 0$$

on $X \setminus Z$ and

$$\sqrt{-1}(\Theta(E) + \text{Id}_E \otimes \Theta(F) - \varepsilon_0 \text{Id}_E \otimes \Theta(L)) \geq_{\text{Nak}} 0$$

on $X \setminus Z$ for some positive real number ε_0 , we have a small positive real number ε_1 such that

$$\sqrt{-1}(\eta \Theta_{(E \otimes F, h_E h_F e^{-\mu(\varphi)})} - \text{Id}_E \otimes \partial \bar{\partial} \eta) \geq_{\text{Nak}} \sqrt{-1}(\text{Id}_E \otimes \eta^{-2} \partial \eta \wedge \bar{\partial} \eta)$$

holds on $V \setminus Z$ for $0 < \varepsilon < \varepsilon_1$, where $\eta = \eta_{\varepsilon,\lambda}$ or $\eta_{\varepsilon,\mu}$.

Proof. By the definitions of λ and μ , $\partial \bar{\partial} \mu(\varphi) = \partial \bar{\partial} \lambda(\varphi) + \partial \bar{\partial} \varphi$, and $\lambda(\varphi)$ and $\mu(\varphi)$ are plurisubharmonic. Note that

$$\begin{aligned} \Theta_{(E \otimes F, h_E h_F e^{-\mu(\varphi)})} &= \Theta(E) + \text{Id}_E \otimes \Theta(F) + \text{Id}_E \otimes \partial \bar{\partial} \mu(\varphi) \\ &= \Theta(E) + \text{Id}_E \otimes \Theta(F) + \text{Id}_E \otimes \partial \bar{\partial} \lambda(\varphi) + \text{Id}_E \otimes \partial \bar{\partial} \varphi, \\ \Theta_{(L, h_L e^{-\mu(\varphi)})} &= \Theta(L) + \partial \bar{\partial} \mu(\varphi), \\ \Theta_{(L, h_L e^{-\lambda(\varphi)})} &= \Theta(L) + \partial \bar{\partial} \lambda(\varphi). \end{aligned}$$

We also note that

$$0 \leq \chi'(\sigma_{\varepsilon,\lambda}) \frac{|s|_{\lambda(\varphi)}^2}{|s|_{\lambda(\varphi)}^2 + \varepsilon} < \frac{7}{4}$$

and

$$0 \leq \chi'(\sigma_{\varepsilon,\mu}) \frac{|s|_{\mu(\varphi)}^2}{|s|_{\mu(\varphi)}^2 + \varepsilon} < \frac{7}{4}$$

by Remark 3.4. If ε_1 is small, then

$$\eta \geq \max \left\{ \frac{7}{4\varepsilon_0}, \frac{7}{4} \right\}$$

since $\eta > \frac{1}{\varepsilon} > \frac{1}{\varepsilon_1}$. Therefore,

$$\sqrt{-1}(\eta \Theta_{(E \otimes F, h_E h_F e^{-\mu(\varphi)})} - \text{Id}_E \otimes \partial \bar{\partial} \eta) \geq_{\text{Nak}} \sqrt{-1}(\text{Id}_E \otimes \eta^{-2} \partial \eta \wedge \bar{\partial} \eta)$$

holds on $V \setminus Z$ for $0 < \varepsilon < \varepsilon_1$ where $\eta = \eta_{\varepsilon,\lambda}$ or $\eta_{\varepsilon,\mu}$. \square

We note that we need no assumptions on the sign of the curvature $\sqrt{-1}\Theta(L)$ in Proposition 3.9. It is a very important remark.

In the next lemma, we obtain the relationship between the Chern connections of (L, h_L) and $(L, h_L e^{-\lambda(\varphi)})$.

Lemma 3.10. *Let $\gamma : [0, \infty) \rightarrow \mathbb{R}$ be any smooth \mathbb{R} -valued function. Then we have the following equation by the definition of the Chern connection:*

$$\begin{aligned} D'_{(L, h_L e^{-\gamma(\varphi)})} &= (H_L e^{-\gamma(\varphi)})^{-1} \partial (H_L e^{-\gamma(\varphi)}) \cdot \\ &= \partial + \partial \log(H_L e^{-\gamma(\varphi)}) \wedge \cdot \\ &= \partial + \partial \log H_L \wedge \cdot - \gamma'(\varphi) \partial \varphi \wedge \cdot \\ &= D'_{(L, h_L)} - \gamma'(\varphi) \partial \varphi \wedge \cdot. \end{aligned}$$

We note that $H_L = \overline{H}_L$ since L is a line bundle.

3.11 (Complete Kähler metrics). There exists a complete Kähler metric g on V since V is weakly 1-complete. Let ω be the fundamental form of g . We note the following well-known lemma (cf. [2, Lemma 5]).

Lemma 3.12. *There exists a quasi-psh function ψ on X such that $\psi = -\infty$ on Z with logarithmic poles along Z and ψ is smooth outside Z .*

Without loss of generality, we can assume that $\psi < -e$ on $V \Subset X$. We put

$$\tilde{\psi} = \frac{1}{\log(-\psi)}.$$

Then $\tilde{\psi}$ is a quasi-psh function on V and $\tilde{\psi} < 1$. Thus, we can take a positive constant α such that

$$\sqrt{-1} \partial \bar{\partial} \tilde{\psi} + \alpha \omega > 0$$

on $V \setminus Z$. Let g' be the Kähler metric on $V \setminus Z$ whose fundamental form is

$$\omega' = \omega + (\sqrt{-1} \partial \bar{\partial} \tilde{\psi} + \alpha \omega).$$

We note that we can check that

$$\omega' \geq \sqrt{-1} \partial \bar{\partial} (\log(\log(-\psi))) \wedge \bar{\partial} (\log(\log(-\psi)))$$

if we choose $\alpha \gg 0$. It is because

$$\partial \bar{\partial} \tilde{\psi} = 2 \frac{\frac{-\partial \psi}{-\psi} \wedge \frac{-\bar{\partial} \psi}{-\bar{\psi}}}{(\log(-\psi))^3} + \frac{\frac{\partial \bar{\partial} \psi}{-\psi}}{(\log(-\psi))^2} + \frac{\frac{-\partial \psi \wedge (-\bar{\partial} \psi)}{(-\psi)^2}}{(\log(-\psi))^2},$$

and

$$\partial (\log(\log(-\psi))) = \frac{\frac{-\partial \psi}{-\psi}}{\log(-\psi)}.$$

Therefore, g' is a *complete* Kähler metric on $V \setminus Z$ by Hopf–Rinow because $\log(\log(-\psi))$ tends to $+\infty$ on Z . For similar arguments, see [13, Section 3]. We fix these Kähler metrics throughout this proof.

3.13 (Key results). The following three propositions are the heart of the proof of Theorem 1.1.

Proposition 3.14. *For every $u \in \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$, $(\bar{\partial}\eta)^*u = 0$ for $\eta = \eta_{\varepsilon,\lambda}$ and $\eta_{\varepsilon,\mu}$. This implies that $\partial\eta \wedge *u = 0$ for $\eta = \eta_{\varepsilon,\lambda}$ and $\eta_{\varepsilon,\mu}$. Thus, we obtain*

$$D'_{(L, h_L e^{-\lambda(\varphi)})^S} \wedge *u = 0 \quad \text{and} \quad D'_{(L, h_L e^{-\mu(\varphi)})^S} \wedge *u = 0.$$

Proof. The definition of $\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$ implies that $\bar{\partial}u = 0$ and $D''^*_{(E \otimes F, h_E h_F e^{-\mu(\varphi)})}u = 0$. By Propositions 2.20, 2.21, and 3.9, we have

$$0 \geq -\|\sqrt{\eta}D'^*u\|^2 \geq \langle \sqrt{-1}\eta^{-2}\partial\eta \wedge \bar{\partial}\eta \wedge u, u \rangle \geq 0.$$

Thus, we have $(\bar{\partial}\eta)^*u = 0$ (cf. Lemma 2.22). Therefore, we obtain that $\partial\eta \wedge *u = 0$ because $(\bar{\partial}\eta)^* = *\partial\eta \wedge *$ by Lemma 2.15. By the definition of η , we obtain $D'_{(L, h_L e^{-\lambda(\varphi)})^S} \wedge *u = 0$ (resp. $D'_{(L, h_L e^{-\mu(\varphi)})^S} \wedge *u = 0$) if $\eta = \eta_{\varepsilon,\lambda}$ (resp. $\eta = \eta_{\varepsilon,\mu}$). \square

Proposition 3.15. *If $D'_{(L, h_L e^{-\lambda(\varphi)})^S} \wedge *u = 0$ and $D'_{(L, h_L e^{-\mu(\varphi)})^S} \wedge *u = 0$, then*

$$D'_{(L, h_L)^S} \wedge *u = 0 \quad \text{and} \quad \partial\varphi \wedge *u = 0.$$

*Therefore, $D'_{(L, h_L e^{-v(\varphi)})^S} \wedge *u = 0$ for any smooth \mathbb{R} -valued function v defined on $[0, \infty)$.*

Proof. We note that $D'_{(L, h_L e^{-\lambda(\varphi)})} = D'_{(L, h_L)} - \lambda'(\varphi)\partial\varphi \wedge \cdot$ and

$$D'_{(L, h_L e^{-\mu(\varphi)})} = D'_{(L, h_L)} - \lambda'(\varphi)\partial\varphi \wedge \cdot - \partial\varphi \wedge \cdot$$

since $\mu(x) = \lambda(x) + x$. \square

Proposition 3.16. *Assume that $D'_{(L, h_L e^{-v(\varphi)})^S} \wedge *u = 0$ for a smooth \mathbb{R} -valued function v defined on $[0, \infty)$. If $D''^*_{(E \otimes F, h_E h_F e^{-\mu(\varphi)})}u = 0$, then we obtain*

$$D''^*_{(E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-v(\varphi)})}(su) = 0.$$

Proof. Let H_E (resp. H_F) be the local matrix representation of h_E (resp. h_F). The condition $D''^*_{(E \otimes F, h_E h_F e^{-\mu(\varphi)})}u = 0$ implies that

$$\bar{\partial}(e^{-\mu(\varphi)} H_E H_F \overline{*u}) = 0.$$

To prove $D''^*_{(E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-v(\varphi)})}(su) = 0$, it is sufficient to check that

$$\bar{\partial}(H_E H_F e^{-\mu(\varphi)-v(\varphi)} H_L \overline{*su}) = 0.$$

We note that

$$\bar{\partial}(H_E H_F e^{-\mu(\varphi)-v(\varphi)} H_L \overline{*su}) = \bar{\partial}(H_L \bar{s} e^{-v(\varphi)}) \wedge e^{-\mu(\varphi)} H_E H_F \overline{*u}$$

by the above condition. The right hand side is zero since $D'_{(L, h_L e^{-v(\varphi)})^S} \wedge *u = 0$. \square

The next theorem is a key result.

Theorem 3.17 (cf. [27, Proposition 3.1]). *For any smooth \mathbb{R} -valued function v defined on $[0, \infty)$ such that $v \geq C$ for some constant C ,*

$$s\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$$

is contained in

$$\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi) - \lambda(\varphi) - v(\varphi)}))_{g'}$$

for every q .

Proof. Let $u \in \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$. Then it is obvious that

$$su \in L_{(2)}^{n,q}(E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi) - \lambda(\varphi) - v(\varphi)})$$

because

$$|s|_{\lambda(\varphi)}^2 < \frac{1}{5} \quad \text{and} \quad 0 < e^{-v(\varphi)} \leq e^{-C}.$$

So, the claim is a direct consequence of Propositions 3.14, 3.15, and 3.16. Note that $\bar{\partial}(su) = 0$ for $u \in \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$ since s is holomorphic and $\bar{\partial}u = 0$. \square

3.18 (Cohomology groups). Before we start the proof of the main theorem, Theorem 1.1, we represent the cohomology groups on V by the objects on $V \setminus Z$.

Definition 3.19 (Space of locally square integrable forms). We define the space of locally (in V) square integrable $E \otimes F$ -valued (n, q) -forms on $V \setminus Z$. It is denoted by

$$L_{\text{loc}, V}^{n,q}(V \setminus Z, E \otimes F) \quad \text{or} \quad L_{\text{loc}, V}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F)).$$

The vector space $L_{\text{loc}, V}^{n,q}(V \setminus Z, E \otimes F)$ is spanned by (n, q) -forms u on $V \setminus Z$ with measurable coefficients such that

$$\int_U |u|_{g', h_E h_F}^2 dV_{\omega'} < \infty$$

for every $U \Subset V$ (not $U \Subset V \setminus Z$), where $|\cdot|_{g', h_E h_F}$ is the pointwise norm with respect to g' and $h_E h_F$. We note the following obvious remark. Let $h : V \rightarrow (0, \infty)$ be a smooth positive function. Then

$$L_{\text{loc}, V}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F)) = L_{\text{loc}, V}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F h)).$$

We can define $L_{\text{loc}, V}^{n,q}(V \setminus Z, E \otimes F \otimes L)$ similarly.

The next lemma is essentially the same as [13, Claim 1], which is more or less known to experts (cf. [30, Proposition 4.6]).

Lemma 3.20. *The following isomorphism holds:*

$$\begin{aligned} H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F)) &\simeq H_{\text{loc}, V}^{n,q}(V \setminus Z, E \otimes F)_{g'} \\ &:= \frac{\text{Ker } \bar{\partial} \cap L_{\text{loc}, V}^{n,q}(V \setminus Z, E \otimes F)}{L_{\text{loc}, V}^{n,q}(V \setminus Z, E \otimes F) \cap \bar{\partial} L_{\text{loc}, V}^{n,q-1}(V \setminus Z, E \otimes F)}. \end{aligned}$$

Sketch of the proof. Let $V = \bigcup_{i \in I} U_i$ be a locally finite Stein cover of V such that each U_i is sufficiently small and $U_i \Subset V$. We denote this cover by $\mathcal{U} = \{U_i\}_{i \in I}$. By Cartan and Leray, we obtain

$$H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F)) \simeq \check{H}^q(\mathcal{U}, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F)),$$

where the right hand side is the Čech cohomology group calculated by \mathcal{U} . By using a partition of unity $\{\rho_i\}_{i \in I}$ associated to \mathcal{U} , we can construct a homomorphism

$$\rho : \check{H}^q(\mathcal{U}, K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \rightarrow H_{\text{loc}, V}^{n, q}(V \setminus Z, E \otimes F)_{g'}.$$

See Remark 3.21 below. We can check that ρ is an isomorphism. Note the following facts: (a) The open set $U_{i_0} \cap \cdots \cap U_{i_k}$ is Stein. So, $U_{i_0} \cap \cdots \cap U_{i_k} \setminus Z$ is a complete Kähler manifold (cf. [1, Théorème 0.2]). Therefore, $E \otimes F$ -valued $\bar{\partial}$ -equations can be solved with suitable L^2 estimates on $U_{i_0} \cap \cdots \cap U_{i_k} \setminus Z$ by Lemma 3.22 below. (b) Let U be an open subset of V . An $E \otimes F$ -valued holomorphic $(n, 0)$ -form on $U \setminus Z$ with a finite L^2 norm can be extended to an $E \otimes F$ -valued holomorphic $(n, 0)$ -form on U (cf. Remark 3.21). \square

Remark 3.21 (cf. [1, Lemme 3.3]). Let u be an $E \otimes F$ -valued (n, q) -form on $V \setminus Z$ with measurable coefficients. Then, we have

$$|u|_{g', h_E h_F}^2 dV_{\omega'} \leq |u|_{g, h_E h_F}^2 dV_{\omega},$$

where $|u|_{g', h_E h_F}$ (resp. $|u|_{g, h_E h_F}$) is the pointwise norm induced by g' and $h_E h_F$ (resp. g and $h_E h_F$) since $g' > g$ on $V \setminus Z$. If u is an $E \otimes F$ -valued $(n, 0)$ -form, then

$$|u|_{g', h_E h_F}^2 dV_{\omega'} = |u|_{g, h_E h_F}^2 dV_{\omega}.$$

The following lemma is [13, Lemma 3.2], which is a reformulation of the classical L^2 -estimates for $\bar{\partial}$ -equations for our purpose.

Lemma 3.22 (L^2 -estimates for $\bar{\partial}$ -equations on complete Kähler manifolds). *Let U be a sufficiently small Stein open set of V . If $u \in L_{(2)}^{n, q}(U \setminus Z, E \otimes F)_{g', h_E h_F}$ with $\bar{\partial}u = 0$, then there exists $v \in L_{(2)}^{n, q-1}(U \setminus Z, E \otimes F)_{g', h_E h_F}$ such that $\bar{\partial}v = u$. Moreover, there exists a positive constant C independent of u such that*

$$\int_{U \setminus Z} |v|_{g', h_E h_F}^2 \leq C \int_{U \setminus Z} |u|_{g', h_E h_F}^2.$$

We note that g' is not a complete Kähler metric on $U \setminus Z$ but $U \setminus Z$ is a complete Kähler manifold (cf. [1, Théorème 0.2]).

By the same arguments, the isomorphism in Lemma 3.20 holds even when we replace $(E \otimes F, h_E h_F)$ with $(E \otimes F \otimes L, h_E h_F h_L)$.

3.23 (Proof of the main theorem, Theorem 1.1). Let us start the proof of Theorem 1.1 (cf. [27]).

Proof of Theorem 1.1. Let u be any $\bar{\partial}$ -closed locally square integrable $E \otimes F$ -valued (n, q) -form on $V \setminus Z$ such that $su = \bar{\partial}v$ for some $v \in L_{\text{loc}, V}^{n, q-1}(V \setminus Z, E \otimes F \otimes L)$. We choose

λ such that $|s|_{\lambda(\varphi)}^2 < \frac{1}{5}$ and $u \in L_{(2)}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\lambda(\varphi)}))$. Since

$$\mu(\varphi) = \lambda(\varphi) + \varphi,$$

it follows that

$$u \in L_{(2)}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)})).$$

By choosing v suitably, we can assume that

$$v \in L_{(2)}^{n,q-1}(V \setminus Z, (E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-v(\varphi)})).$$

In particular,

$$v \in L_{(2)}^{n,q-1}(V \setminus Z, (E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-\lambda(\varphi)-v(\varphi)})).$$

Let Pu be the orthogonal projection of u to $\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}$. We note that

$$\begin{aligned} & L_{(2)}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)})) \\ &= \overline{\text{Im } \bar{\partial}} \oplus \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'} \oplus \overline{\text{Im } D_{(E \otimes F, h_E h_F e^{-\mu(\varphi)})}''*}, \end{aligned}$$

and

$$\text{Ker } \bar{\partial} = \overline{\text{Im } \bar{\partial}} \oplus \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))_{g'}.$$

Here,

$$\overline{\text{Im } \bar{\partial}} \quad (\text{resp. } \overline{\text{Im } D_{(E \otimes F, h_E h_F e^{-\mu(\varphi)})}''*})$$

denotes the closure of

$$\bar{\partial} C_0^{n,q-1}(V \setminus Z, E \otimes F) \quad (\text{resp. } D_{(E \otimes F, h_E h_F e^{-\mu(\varphi)})}''* C_0^{n,q+1}(V \setminus Z, E \otimes F))$$

in $L_{(2)}^{n,q}(V \setminus Z, (E \otimes F, h_E h_F e^{-\mu(\varphi)}))$. Note that the fixed Kähler metric g' is complete. It follows that $u - Pu$ is in the closure of the image of $\bar{\partial}$. Thus, so is $s(u - Pu)$ since s is holomorphic. On the other hand,

$$sPu \in \mathcal{H}^{n,q}(V \setminus Z, (E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-\lambda(\varphi)-v(\varphi)}))_{g'}$$

by Theorem 3.17. So, sPu coincides with the orthogonal projection of su to

$$\mathcal{H}^{n,q}(V \setminus Z, (E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-\lambda(\varphi)-v(\varphi)}))_{g'},$$

which must be equal to zero since $v \in L_{(2)}^{n,q-1}(V \setminus Z, (E \otimes F \otimes L, h_E h_F h_L e^{-\mu(\varphi)-\lambda(\varphi)-v(\varphi)}))$. Therefore, $Pu = 0$. Since $H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F))$ is a separated topological vector space (cf. [28, Lemma II.1.]) and

$$u \in \overline{\text{Im } \bar{\partial}},$$

there exists

$$w \in L_{\text{loc},V}^{n,q-1}(V \setminus Z, E \otimes F)$$

such that $u = \bar{\partial}w$ (cf. [30, Proposition 4.6] and [13, Claim 1]). This means that u represents zero in $H^q(V, K_V \otimes E \otimes F \otimes \mathcal{J}(h_F))$. \square

4. Corollaries and applications

In this section, we discuss the proofs of corollaries in Section 1 and some applications of Theorem 1.1.

First, we give a proof of Corollary 1.2, which is obvious if we apply Theorem 1.1 for

$$L = \mathcal{O}_X \simeq f^* \mathcal{O}_Y.$$

Proof of Corollary 1.2. The statement is local. So, we can assume that Y is Stein. Let $s \in H^0(Y, \mathcal{O}_Y)$ be an arbitrary nonzero section. By Theorem 1.1,

$$\times s : R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \rightarrow R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F))$$

is injective for every $q \geq 0$. Thus, $R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F))$ is torsion-free for every $q \geq 0$. \square

The following proposition is a slight generalization of Theorem 1.1.

Proposition 4.1. *In Theorem 1.1, we can weaken the assumption that X is Kähler as follows. For any point $P \in Y$, there exist an open neighborhood U of P and a proper bimeromorphic morphism $g : W \rightarrow V := f^{-1}(U)$ from a Kähler manifold W .*

Sketch of the proof. We put $Z' = g^{-1}(Z \cap V) \subset W$. We can apply Corollary 1.2 to $g : W \rightarrow V$, Z' , (g^*E, g^*h_E) , and (g^*F, g^*h_F) . Then we obtain

$$R^q g_*(K_W \otimes g^*E \otimes g^*F \otimes \mathcal{J}(g^*h_F)) = 0$$

for every $q > 0$ and it is well known (and easy to check) that

$$g_*(K_W \otimes g^*E \otimes g^*F \otimes \mathcal{J}(g^*h_F)) \simeq K_V \otimes E \otimes F \otimes \mathcal{J}(h_F)$$

(cf. [4, (5.8) Proposition]). Therefore, by Leray's spectral sequence,

$$R^q (f \circ g)_*(K_W \otimes g^*E \otimes g^*F \otimes \mathcal{J}(g^*h_F)) \simeq R^q f_*(K_V \otimes E \otimes F \otimes \mathcal{J}(h_F))$$

for every $q \geq 0$. Apply Theorem 1.1 to $f \circ g : W \rightarrow U$, Z' , (g^*E, g^*h_E) , (g^*L, g^*h_L) , and (g^*F, g^*h_F) . Then we obtain that

$$\times s : R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \rightarrow R^q f_*(K_X \otimes E \otimes F \otimes \mathcal{J}(h_F) \otimes L)$$

is injective for every $q \geq 0$ by the above isomorphisms. \square

The next result is related to the main theorem in [24].

Corollary 4.2. *Let $f : X \rightarrow \Delta$ be a smooth projective surjective morphism from a Kähler manifold X to a disk Δ and let E be a Nakano semi-positive vector bundle on X . Assume that there exists $D \in |K_{X_0}^{\otimes l}|$ such that $\mathcal{J}_{X_0}(cD) \simeq \mathcal{O}_{X_0}$ for every $0 \leq c < 1$, where $0 \in Y$ and $X_0 = f^{-1}(0)$. Then there exists an open set $U \subset \Delta$ such that $0 \in U$ and $R^q f_*(K_X^{\otimes m} \otimes E)$ is locally free on U for every $q \geq 0$ and $1 \leq m \leq l$. Equivalently, $\dim_{\mathbb{C}} H^q(X_t, K_{X_t}^{\otimes m} \otimes E)$ is constant for every $q \geq 0$ by the base change theorem, where $t \in U$ and $X_t = f^{-1}(t)$.*

Proof. In this proof, we shrink Δ without mentioning it for simplicity of notation. Let $s_0 \in H^0(X_0, K_{X_0}^{\otimes l})$ such that

$$D = (s_0 = 0).$$

By Siu's extension theorem (see [29, Theorem 0.1]), there exists $s \in H^0(X, K_X^{\otimes l})$ such that $s|_{X_0} = s_0$. We consider the singular hermitian metric

$$h_{K_X^{\otimes(m-1)}} = \left(\frac{1}{|s|^2} \right)^{\frac{m-1}{l}}$$

of $K_X^{\otimes(m-1)}$. By the Ohsawa–Takegoshi L^2 -extension theorem and the assumption

$$\mathcal{J}_{X_0}(cD) \simeq \mathcal{O}_{X_0} \quad \text{for } 0 \leq c < 1,$$

we obtain that $\mathcal{J}_X(h_{K_X^{\otimes(m-1)}}) \simeq \mathcal{O}_X$ in a neighborhood of X_0 . Therefore, we obtain that

$$R^q f_*(K_X^{\otimes m} \otimes E) = R^q f_*(K_X \otimes K_X^{\otimes(m-1)} \otimes E \otimes \mathcal{J}_X(h_{K_X^{\otimes(m-1)}}))$$

is locally free by Corollary 1.2. □

Let us start the proof of the Kawamata–Viehweg–Nadel type vanishing theorem: Corollary 1.3.

Proof of Corollary 1.3. Let P be a point of Y . The problem is local. So we repeatedly shrink Y around the point P without mentioning it explicitly. Since \mathcal{M} is f -big, we have a bimeromorphic map $\Phi : X \dashrightarrow X' \subset Y \times \mathbb{P}^N$ over Y . By applying Hironaka's Chow lemma (cf. [19, Corollary 2]), we can construct a bimeromorphic map $\varphi : Z \rightarrow X$ from a complex manifold such that $f \circ \varphi : Z \rightarrow Y$ is projective. It is easy to see that

$$\varphi_* \left(K_Z \otimes \varphi^* E \otimes \varphi^* \mathcal{L} \otimes \mathcal{J} \left(\frac{1}{m} \varphi^* D \right) \right) \simeq K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J} \left(\frac{1}{m} D \right).$$

by the definition of \mathcal{J} (cf. [23, Theorem 9.2.33]).

Claim. *We have*

$$R^q \varphi_* \left(K_Z \otimes \varphi^* E \otimes \varphi^* \mathcal{L} \otimes \mathcal{J} \left(\frac{1}{m} \varphi^* D \right) \right) = 0$$

for every $q > 0$

Proof of Claim. The problem is local. So we can shrink X and assume that \mathcal{M} is trivial. In particular, $(\varphi^* \mathcal{L})^{\otimes m} \simeq \mathcal{O}_Z(\varphi^* D)$. Thus, by Corollary 1.2, we obtain that

$$R^q \varphi_* \left(K_Z \otimes \varphi^* E \otimes \varphi^* \mathcal{L} \otimes \mathcal{J} \left(\frac{1}{m} \varphi^* D \right) \right)$$

is torsion-free for every q . Thus, $R^q \varphi_*(K_Z \otimes \varphi^* E \otimes \varphi^* \mathcal{L} \otimes \mathcal{J}(\frac{1}{m} \varphi^* D)) = 0$ for every $q > 0$ since φ is bimeromorphic. □

Therefore, by replacing X with Z , we can assume that $f : X \rightarrow Y$ is projective. By Kodaira's lemma, we can write $\mathcal{L}^{\otimes a} \simeq \mathcal{A} \otimes \mathcal{O}_X(G)$ where a is a positive integer, \mathcal{A} is an

f -ample line bundle on X , and G is an effective Cartier divisor on X . Then

$$\mathcal{L}^{\otimes(mb+a)} \simeq (\mathcal{M}^{\otimes b} \otimes \mathcal{A}) \otimes \mathcal{O}_X(G + bD).$$

Note that $\mathcal{M}^{\otimes b} \otimes \mathcal{A}$ is f -ample and that $\mathcal{J}(\frac{G+bD}{mb+a}) = \mathcal{J}(\frac{1}{m}D)$ if $b \gg 0$. Therefore, we can further assume that \mathcal{M} is f -ample. Then, by Theorem 1.1, we can construct inclusions

$$R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) \subset R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J} \otimes \mathcal{M}^{\otimes k})$$

for every $k > 0$ with $|\mathcal{M}^{\otimes k}| \neq \emptyset$. Thus, by Serre's vanishing theorem (cf. [26, p. 25, Remark (4)]), we obtain $R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0$ for every $q > 0$. \square

Proof of Corollary 1.4. Note that $m\{H\}$ is Cartier. We have $m^\top H^\top \sim mH + m\{-H\}$. We put $E = \mathcal{O}_X$, $\mathcal{L} = \mathcal{O}_X(\top H^\top)$, $\mathcal{M} = \mathcal{O}_X(mH)$, and $D = m\{-H\}$, and apply Corollary 1.3. Then we obtain

$$R^q f_*(K_X \otimes \mathcal{O}_X(\top H^\top)) = 0 \quad \text{for every } q > 0.$$

We note that

$$\mathcal{J}(\{-H\}) = \mathcal{O}_X$$

since $\text{Supp}\{H\}$ is a normal crossing divisor (cf. [23, Lemma 9.3.44]). \square

The proof of Corollary 1.5 is a routine work. So we only sketch the proof here. For details, see, for example, the proofs of [11, Theorem 1.1 (ii)] and [12, Theorem 6.3 (ii)].

Sketch of the proof of Corollary 1.5. We can repeatedly shrink Z without mentioning it. By Kodaira's lemma, we can assume that \mathcal{N} is g -ample. Let A be a general smooth sufficiently g -ample Cartier divisor on Y . We put $B = f^*A$. We consider the following short exact sequence:

$$0 \rightarrow K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J} \rightarrow K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J} \rightarrow K_B \otimes E|_B \otimes \mathcal{L}|_B \otimes \mathcal{J}|_B \rightarrow 0.$$

Since A is general, $\mathcal{J}|_B = \mathcal{J}(\frac{1}{m}D|_B)$, and

$$\begin{aligned} 0 \rightarrow R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) &\rightarrow R^q f_*(K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J}) \\ &\rightarrow R^q f_*(K_B \otimes E|_B \otimes \mathcal{L}|_B \otimes \mathcal{J}|_B) \rightarrow 0 \end{aligned}$$

is exact for every q . By taking the long exact sequence, we obtain

$$R^p g_* R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0$$

for every $p \geq 2$ and every $q \geq 0$ because A is sufficiently g -ample and the induction on dimension. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} R^1 g_* R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) & \xrightarrow{\alpha} & R^{1+q}(g \circ f)_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) \\ \downarrow & & \downarrow \beta \\ R^1 g_* R^q f_*(K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J}) & \longrightarrow & R^{1+q}(g \circ f)_*(K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J}). \end{array}$$

Note that α is injective by the above vanishing result and that β is injective by Theorem 1.1. Since A is sufficiently g -ample, we have $R^1 g_* R^q f_*(K_X \otimes E \otimes \mathcal{O}_X(B) \otimes \mathcal{L} \otimes \mathcal{J}) = 0$. Thus, $R^1 g_* R^q f_*(K_X \otimes E \otimes \mathcal{L} \otimes \mathcal{J}) = 0$ for every $q \geq 0$. So we finish the proof. \square

5. Examples: nef, semi-positive, and semi-ample line bundles

In this section, we collect some examples of nef, semi-positive, and semi-ample line bundles. These examples help us understand our results in [13] and this paper. We think that it is very important to understand the differences in the notion of semi-ample, semi-positive, and nef line bundles.

First, we recall the following well-known example. It implies that there exists a nef line bundle that has no smooth hermitian metrics with semi-positive curvature.

Example 5.1 (cf. [5, Example 1.7]). Let C be an elliptic curve and let \mathcal{E} be the rank two vector bundle on C which is defined by the unique non-splitting extension

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow \mathcal{O}_C \rightarrow 0.$$

We consider the ruled surface $X := \mathbb{P}_C(\mathcal{E})$. On that surface there is a unique section

$$D := \mathbb{P}_C(\mathcal{O}_C) \subset X$$

of $X \rightarrow C$ such that $\mathcal{O}_D(D) \simeq \mathcal{O}_D$ and $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is a nef line bundle (cf. [18, V, Proposition 2.8]). It is not difficult to see that

$$H^1(X, K_X \otimes \mathcal{O}_X(2D)) \rightarrow H^1(X, K_X \otimes \mathcal{O}_X(3D))$$

is a zero map, $H^1(X, K_X \otimes \mathcal{O}_X(2D)) \simeq \mathbb{C}$, and $H^1(X, K_X \otimes \mathcal{O}_X(3D)) \simeq \mathbb{C}$. We note that $K_X \sim \mathcal{O}_X(-2D)$. Therefore, $\mathcal{O}_X(D)$ has no smooth hermitian metrics with semi-positive curvature by Enoki's injectivity theorem (see [6, Theorem 0.2], [13, Corollary 1.4]). Note that

$$\kappa(X, \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)) = 0 \quad \text{and} \quad \nu(X, \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)) = 1.$$

We also note that Kollár's injectivity theorem implies nothing since $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is not semi-ample.

The next one is an example of nef and big line bundles that have no smooth hermitian metrics with semi-positive curvature. The author learned the following construction from Dano Kim.

Example 5.2. We use the same notation as in Example 5.1. Let $P \in C$ be a closed point. We put $\mathcal{F} := \mathcal{E} \oplus \mathcal{O}_C(P)$ and $Y := \mathbb{P}_C(\mathcal{F})$. Then it is easy to see that $\mathcal{L} := \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$ is nef and big (cf. [23, Example 6.1.23]). Since $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ has no smooth hermitian metrics with semi-positive curvature, neither has \mathcal{L} . In this case, $H^i(Y, K_Y \otimes \mathcal{L}^{\otimes k}) = 0$ for $i > 0$ and every $k \geq 1$ by the Kawamata–Viehweg vanishing theorem.

Let us recall some examples of semi-positive line bundles that are not semi-ample.

Example 5.3 (cf. [3, p. 145]). Let C be a smooth projective curve with genus $g(C) \geq 1$. Let $L \in \text{Pic}^0(C)$ be non-torsion. We put

$$\mathcal{E} := \mathcal{O}_C \oplus L \quad \text{and} \quad X := \mathbb{P}_C(\mathcal{E}).$$

Then $\mathcal{L} := \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ is semi-positive, but not semi-ample. We note that $\kappa(X, \mathcal{L}) = 0$ since $H^0(X, \mathcal{L}^{\otimes k}) = H^0(C, S^k(\mathcal{E})) = H^0(C, \mathcal{O}_C) = \mathbb{C}$ for every $k \geq 0$, where $S^k(\mathcal{E})$ is the k -th

symmetric product of \mathcal{E} . We can easily check that

$$K_X = \pi^*(K_C \otimes \det \mathcal{E}) \otimes \mathcal{L}^{\otimes(-2)} = \pi^*(K_C \otimes L) \otimes \mathcal{L}^{\otimes(-2)},$$

where $\pi : X \rightarrow C$ is the projection. Let m be an integer with $m \geq 2$. Then

$$\begin{aligned} H^i(X, K_X \otimes \mathcal{L}^{\otimes m}) &= H^i(X, \pi^*(K_C \otimes L) \otimes \mathcal{L}^{\otimes(m-2)}) \\ &= \bigoplus_{k=1}^{m-1} H^i(C, K_C \otimes L^{\otimes k}). \end{aligned}$$

Thus,

$$\begin{aligned} h^0(X, K_X \otimes \mathcal{L}^{\otimes m}) &= (m-1)(g-1), \\ h^1(X, K_X \otimes \mathcal{L}^{\otimes m}) &= 0, \\ h^2(X, K_X \otimes \mathcal{L}^{\otimes m}) &= 0 \end{aligned}$$

for $m \geq 2$. So, we obtain no interesting results from injectivity theorems. Note that

$$H^2(X, K_X \otimes \mathcal{L}^{\otimes m}) = 0 \quad \text{for } m \geq 1$$

also follows from the Kawamata–Viehweg vanishing theorem as $\nu(X, \mathcal{L}) = 1$ and $\dim X = 2$.

Example 5.4 (Cutkosky). We use the same notation as in Example 5.3. Let $P \in C$ be a closed point. We put

$$\mathcal{F} := \mathcal{O}_C(P) \oplus L \quad \text{and} \quad Y := \mathbb{P}_C(\mathcal{F}).$$

Then it is easy to see that $\mathcal{M} := \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$ is big and semi-positive, but not semi-ample. We note that $\bigoplus_{m \geq 0} H^0(Y, \mathcal{M}^{\otimes m})$ is not finitely generated. For details, see, for example, [22, Example 2.3.3].

The following example shows the difference between Enoki's injectivity theorem and Kollár's one.

Example 5.5. Let C be a smooth projective curve with genus $g(C) = g \geq 1$. Suppose $L \in \text{Pic}^0(C)$ is non-torsion. We put $\mathcal{E} := \mathcal{O}_C \oplus L \oplus L^{-1}$, $X := \mathbb{P}_C(\mathcal{E})$, and $\mathcal{L} := \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. It is obvious that \mathcal{E} has a smooth hermitian metric whose curvature is Nakano semi-positive. Thus, \mathcal{L} is semi-positive since \mathcal{L} is a quotient line bundle of $\pi^*\mathcal{E}$, where $\pi : X \rightarrow C$ is the projection. In particular, \mathcal{L} is nef. On the other hand, it is not difficult to see that \mathcal{L} is not semi-ample. We have

$$K_X = \pi^*(K_C \otimes \det \mathcal{E}) \otimes \mathcal{L}^{\otimes(-3)} = \pi^*K_C \otimes \mathcal{L}^{\otimes(-3)}.$$

We can easily check that

$$S^m(\mathcal{E}) = \bigoplus_{\substack{0 \leq a+b \leq m \\ a, b \geq 0}} L^{a-b}.$$

Note that the rank of $S^m(\mathcal{E})$ is $\frac{1}{2}(m+2)(m+1)$. Let m be an integer with $m \geq 3$. Then it is easy to see that

$$H^i(X, K_X \otimes \mathcal{L}^{\otimes m}) = H^i(X, \pi^*K_C \otimes \mathcal{L}^{\otimes(m-3)}) = H^i(C, K_C \otimes S^{m-3}(\mathcal{E})).$$

for all i . We need the following obvious lemma.

Lemma 5.6. *We have $h^0(C, K_C) = g$ and $h^1(C, K_C) = 1$. Moreover,*

$$h^0(C, K_C \otimes L^{\otimes k}) = g - 1 \quad \text{and} \quad h^1(C, K_C \otimes L^{\otimes k}) = 0 \quad \text{for } k \neq 0.$$

Therefore, we obtain

$$\begin{aligned} H^3(X, K_X \otimes \mathcal{L}^{\otimes m}) &= H^2(X, K_X \otimes \mathcal{L}^{\otimes m}) = 0, \\ h^1(X, K_X \otimes \mathcal{L}^{\otimes m}) &= \lfloor \frac{m-3}{2} \rfloor + 1 = \lfloor \frac{m-1}{2} \rfloor, \end{aligned}$$

and

$$\begin{aligned} h^0(X, K_X \otimes \mathcal{L}^{\otimes m}) &= g \lfloor \frac{m-1}{2} \rfloor + (g-1) \left(\frac{(m+2)(m+1)}{2} - \lfloor \frac{m-1}{2} \rfloor \right) \\ &= \lfloor \frac{m-1}{2} \rfloor + (g-1) \frac{(m+2)(m+1)}{2}. \end{aligned}$$

On the other hand, $h^0(X, \mathcal{L}^{\otimes k}) = h^0(C, S^k(\mathcal{E})) = \lfloor \frac{k}{2} \rfloor + 1$ for $k \geq 0$. Let $s \in |\mathcal{L}^{\otimes k}|$ be a non-zero holomorphic section of $\mathcal{L}^{\otimes k}$ for $k \geq 0$. Then

$$\times s : H^1(X, K_X \otimes \mathcal{L}^{\otimes m}) \rightarrow H^1(X, K_X \otimes \mathcal{L}^{\otimes(m+k)})$$

is injective by Enoki's injectivity theorem (cf. [6, Theorem 0.2], [13, Corollary 1.4]). Note that $h^1(X, K_X \otimes \mathcal{L}^{\otimes m}) = \lfloor \frac{m-1}{2} \rfloor$ and $h^1(X, K_X \otimes \mathcal{L}^{\otimes(m+k)}) = \lfloor \frac{m+k-1}{2} \rfloor$. We have $\kappa(X, \mathcal{L}) = 1$ by the above calculation. Since $\mathcal{L}^2 \cdot F = 1$, where F is a fiber of $\pi : X \rightarrow C$, $\nu(X, \mathcal{L}) = 2$. Thus, the nef line bundle \mathcal{L} is not abundant. So, we think that there are no algebraic proofs for the above injectivity theorem. Note that $H^3(X, K_X \otimes \mathcal{L}^{\otimes m}) = H^2(X, K_X \otimes \mathcal{L}^{\otimes m}) = 0$ for $m \geq 1$ follows from the Kawamata–Viehweg vanishing theorem since $\nu(X, \mathcal{L}) = 2$ and $\dim X = 3$.

The following two examples are famous ones due to Mumford and Ramanujam.

Example 5.7 (Mumford). Let us recall the construction of Mumford's example (see [17, Example 10.6]). We use the same notation as in [17, Example 10.6]. Let C be a smooth projective curve of genus $g \geq 2$ over \mathbb{C} . Then there exists a stable vector bundle E of rank two and $\deg E = 0$ such that its symmetric powers $S^m(E)$ are stable for all $m \geq 1$. We consider the ruled surface $X := \mathbb{P}_C(E)$. Let D be the divisor corresponding to $\mathcal{O}_X(1)$. Since E is a unitary flat vector bundle, $\mathcal{L} := \mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}_C(E)}(1)$ is semi-positive by $\pi^*E \rightarrow \mathcal{L} \rightarrow 0$, where $\pi : X \rightarrow C$ is the projection. We know that

$$H^0(X, \mathcal{L}^{\otimes m}) = H^0(C, S^m(E)) = 0$$

since $S^m(E)$ is stable and $c_1(S^m(E)) = 0$ for every $m \geq 1$. Thus, $\kappa(X, \mathcal{L}) = -\infty$. On the other hand, $\mathcal{L} \cdot \mathcal{L} = 0$ and $\mathcal{L} \cdot C' > 0$ for every curve C' on X . Then $\nu(X, \mathcal{L}) = 1$.

Example 5.8 (Ramanujam). Let us recall the construction of Ramanujam's example (see [17, Example 10.8]). We use the same notation as in [17, Example 10.8]. Let X be the ruled surface obtained in Example 5.7. We assume that D is the divisor given in [17, Example 10.6] (see Example 5.7 above). Let H be an effective ample divisor on X . We define $\bar{X} := \mathbb{P}_X(\mathcal{O}_X(D - H) \oplus \mathcal{O}_X)$, and let $\pi : \bar{X} \rightarrow X$ be the projection. Let X_0 be the section of π corresponding to $\mathcal{O}_X(D - H) \oplus \mathcal{O}_X \rightarrow \mathcal{O}_X(D - H) \rightarrow 0$ and $\bar{D} := X_0 + \pi^*H$. We put $\mathcal{M} := \mathcal{O}_{\bar{X}}(\bar{D})$. We write $\mathcal{O}_{\bar{X}}(1) = \mathcal{O}_{\mathbb{P}_X(\mathcal{O}_X(D-H) \oplus \mathcal{O}_X)}(1)$. Then $\mathcal{O}_{\bar{X}}(1) \simeq \mathcal{O}_{\bar{X}}(X_0)$. There-

fore, $\mathcal{M} \simeq \mathcal{O}_{\bar{X}}(1) \otimes \pi^* \mathcal{O}_X(H)$ and $\pi^*(\mathcal{O}_X(D) \oplus \mathcal{O}_X(H)) \rightarrow \mathcal{M} \rightarrow 0$. Thus, it is easy to see that \mathcal{M} is semi-positive, nef and big. By the construction, $\mathcal{M} \cdot C' > 0$ for every curve C' on \bar{X} . However, \mathcal{M} is not semi-ample since $\mathcal{M}|_{X_0} \simeq \mathcal{O}_{X_0}(D)$ does not have sections on X_0 . In particular, $\bigoplus_{m \geq 0} H^0(\bar{X}, \mathcal{O}_{\bar{X}}(m\bar{D}))$ is not finitely generated.

The author learned the following construction from the referee.

Example 5.9. Let $X = \mathbb{P}_C(\mathcal{E}) \rightarrow C$ be as in Example 5.1. Let A be a very ample divisor on C . We take a smooth member $B \in |2A|$ and take a double cover $\tilde{C} \rightarrow C$ by $B \sim 2A$. We consider the base change diagram

$$\begin{array}{ccc} X & \xleftarrow{\varphi} & \tilde{X} \\ \downarrow & & \downarrow \\ C & \xleftarrow{\quad} & \tilde{C} \end{array}$$

and $\mathcal{A} := \varphi^* \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1) \simeq \mathcal{O}_{\tilde{X}}(\varphi^* D)$. The natural map

$$\alpha : H^1(\tilde{X}, K_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(2\varphi^* D)) \rightarrow H^1(\tilde{X}, K_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(3\varphi^* D))$$

contains

$$\beta : H^1(X, K_X \otimes \mathcal{O}_X(2D)) \rightarrow H^1(X, K_X \otimes \mathcal{O}_X(3D))$$

as a direct summand by the construction. Since β is zero (see Example 5.1), α is not injective. By Enoki's injectivity theorem (cf. [6, Theorem 0.2], [13, Corollary 1.4]), \mathcal{A} is not semi-positive. We note that \tilde{C} is a smooth projective curve of genus $g \geq 2$. Then there exists a stable vector bundle F of rank two and $\deg F = 0$ such that its symmetric powers $S^m(F)$ are stable for all $m \geq 1$ (cf. Example 5.7). We put $Y = \mathbb{P}_{\tilde{C}}(F)$ and $\mathcal{B} = \mathcal{O}_{\mathbb{P}_{\tilde{C}}(F)}(1)$. We take the fiber product

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{p_1} & V = \tilde{X} \times_{\tilde{C}} Y \\ \downarrow & & \downarrow p_2 \\ \tilde{C} & \xleftarrow{\quad} & Y \end{array}$$

and put $\mathcal{M} = p_1^* \mathcal{A} \otimes p_2^* \mathcal{B}$. Note that V is a smooth projective variety. Then it is easy to see that $\mathcal{M} \cdot C' > 0$ for every curve C' on V and that $\kappa(V, \mathcal{M}) = -\infty$. We do not know whether \mathcal{M} is semi-positive or not.

We close this paper with a question.

Question 5.10. In Example 5.9, are there any smooth hermitian metrics on \mathcal{M} with semi-positive curvature?

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